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# Noise of Mode-Locked Lasers

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**Abstract**—A theory of noise in mode-locked lasers is developed that applies to additive pulse mode-locked and Kerr lens mode-locked systems. Equations of motion are derived for pulse energy, carrier linewidth, frequency pulling, and timing jitter. The effect of gain fluctuations, mirror vibrations, and index fluctuations are determined. Measurements that can determine all four fluctuation spectra are described. Experimental data in the literature are compared with theory.

## I. INTRODUCTION

IN recent years, many solid-state lasers have been mode locked successfully, using the additive pulse mode locking (APM) principle [1]–[6]. It has been shown that, under appropriate approximations, the theory of APM is congruent with that of saturable absorber mode locking, supplemented by self-phase modulation (SPM) and group velocity dispersion (GVD) [7]. Another system of passive mode locking has been dubbed Kerr lens mode locking [8]–[10] (KLM). It utilizes self-focusing to produce an intensity dependent loss. It is described by the same theory.

Some of the systems are already incorporated in commercial products. As time proceeds, different schemes will be selected according to their ability to produce stable short pulses with the required duration and intensity. It is likely that the susceptibility to environmental noise will differ among the different systems. One purpose of this paper is to predict the noise characteristics as governed by specified pump, mirror-position, and beam axis fluctuations. Another purpose is more fundamental. To the authors' knowledge, an inclusive theory of noise in mode-locked systems with saturable absorber action, SPM and GVD does not exist. The reason for this is that perturbation theories developed for Hamiltonian systems do not apply; the system is not self-adjoint. In this paper we develop an analytic theory of adiabatic pulse evolution in the presence of noise. We take advantage of the fact that the shortest pulses are achieved when they are transform limited and secant hyperbolic in temporal shape. If GVD and SPM are important, they are perturbed solitons. In

this limit, a perturbation theory is feasible, closely related to that of soliton perturbation theory [11]. Yet even in the case when the simple saturable absorber mode-locking equations apply [12], the perturbation equations can be used, qualitatively, with proper interpretation of the parameters.

In Section II we review the master equation of saturable absorber mode locking in the presence of (negative) GVD and summarize its solution. Section III investigates a linearized equation for the perturbation from the steady-state solution as driven by, as yet, unspecified noise sources. The perturbed pulse is described by four parameters, the amplitude perturbation, and the perturbations of phase, frequency, and timing. In addition, there is a continuum that is not part of the perturbed pulse. In order to project out the amplitudes of the perturbation from a specified complex amplitude function, one uses solutions of the adjoint system which are orthogonal to the solutions of the linearized equation of motion (Section IV). Using the orthogonality properties, equations of motion are written for the four pulse perturbation parameters. These are linear coupled first order differential equations in time, driven by noise sources.

The analysis ignores the coupling to the continuum. This is legitimate as long as the continuum does not react back appreciably. One may develop plausibility arguments that the approximation is valid. Section V identifies the noise sources starting from different realizations of APM systems and KLM systems and expresses the noise sources in terms of physical perturbation spectra. Section VI evaluates the theoretical spectra and the correlation functions of the four observables. Section VII determines ways of measuring the spectra of each of the four pulse parameters. Section VIII compares experimental data with theory.

## II. THE MASTER EQUATION AND ITS SOLUTION

The master equation for the nonlinear pulse evolution as a function of the slow time variable  $T$  (expressing the evolution over many cavity roundtrips) is [7]

$$T_R \frac{\partial}{\partial T} a = \left[ -l + g \left( 1 - \frac{1}{\Omega_R} \frac{\partial}{\partial t} + \frac{1}{\Omega_R^2} \frac{\partial^2}{\partial t^2} \right) + jD \frac{\partial^2}{\partial t^2} + (\gamma - j\delta)|a|^2 \right] a + T_R S(t, T). \quad (1)$$

Here,  $T_R$  is the round-trip time;  $a(T, t)$  is the (electric field) amplitude in the resonator, a function of the short-term time variable  $t$  as well as of  $T$ , a time variable on the

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scale of many cavity round-trip times;  $l$  is the incremental loss,  $g$  the incremental gain upon one pass,  $\Omega_g$  is the gain bandwidth;  $\gamma$  is the effective saturable absorber action that may be produced by APM action,  $\delta$  is the Kerr phase modulation coefficient,  $D$  is the group velocity dispersion coefficient;  $S(t, T)$  is the noise source [10]. The operator multiplying the gain  $g$  expresses the dependence of the group velocity upon the gain and the "diffusion" of a pulse passing the gain medium because of the finite gain bandwidth.

The gain is assumed to have a long relaxation time, so that the change of gain upon one pass can be ignored. The gain is saturated by many pulses in succession. It is related to the small-signal gain  $g_o$  by

$$g = g_o \frac{1}{1 + \frac{1}{P_s T_R} \int dt |a|^2} \quad (2)$$

where  $P_s$  is the saturation power. The saturation law (2) neglects dynamics related to gain relaxation. Thus effects that depend critically on the gain relaxation time are not considered. The amplitude  $a(T, t)$  is so normalized that  $|a|^2$  is equal to the power. The solution of (1) is [12], [13]

$$a(t, T) = A_o \operatorname{sech} \left[ \frac{1}{\tau} (t - t_o) \right]^{(1+j\beta)} \exp \left( j\psi \frac{T}{T_R} + j\theta \right) \quad (3)$$

where

$$-j\psi + g - l + \frac{(1+j\beta)^2}{\tau^2} \left[ \frac{g}{\Omega_g^2} + jD \right] = 0 \quad (4)$$

$$\frac{1}{\tau^2} \left[ \frac{g}{\Omega_g^2} + jD \right] (2 + 3j\beta - \beta^2) = (\gamma - j\delta) A_o^2 \quad (5)$$

and

$$t_o = \frac{g}{\Omega_g T_R} T. \quad (6)$$

Here,  $\beta$  is the chirp parameter;  $\theta$  is the phase of the pulse, an arbitrary parameter in the noise-free solution, yet a driven quantity when noise is introduced;  $\psi$  is the phase shift produced per pass, and determined from the solution of (4). Upon multiple passes, this phase shift per pass translates into a net frequency shift.

Equation (5) shows that the area  $A_o \tau$  of the absolute magnitude of the pulse amplitude is fixed for a fixed set of parameters of the system. The perturbation analysis can proceed from two starting points. One is the original equation of fast saturable absorber mode locking, in the absence of GVD and SPM [12], [14] with  $\delta = D = 0$ . One may find a set of eigenfunctions of the inphase and quadrature perturbations and expand the noise sources in this orthogonal set of eigenfunctions. Each of the amplitudes of the expansion obeys a first-order differential equation in  $T$ . SPM and GVD can then be treated as additional perturbations. We shall not pursue this approach

here, because systems with genuine "physical" fast saturable absorbers are not of practical importance, because of the difficulty in finding suitable absorbing materials. Of much greater practical importance have become systems that construct artificial saturable absorbers by APM or KLM action. Such systems, generally, have appreciable SPM action compensated by negative GVD. For this reason, the pulses are soliton-like, yet not solitons; they are solitary pulses. Contrary to regular soliton perturbations, amplitude and frequency perturbations of mode-locked solitary pulses do not persist, but "shed" some of their energy into the continuum. This is a consequence of the gain saturation and gain-bandwidth limitation not encountered with solitons (nonlinear Schrödinger equation, NLSE). The feedback of the continuum radiation back into the pulse is of second order in this coupling to the continuum and will be neglected. This approximation will be confirmed by computer simulation later on in the paper.

The chirp parameter  $\beta$  is zero when the gain dispersion and APM action are related to the SPM and GVD by

$$\frac{g/\Omega_g^2}{-D} = \frac{\gamma}{\delta} \equiv \mu. \quad (7)$$

Since  $\delta > 0$  for the usual Kerr effect and  $\gamma > 0$  for stability, this is the case of negative GVD. This adjustment leads to transform limited pulses. Henceforth we shall concentrate on this special case. One has

$$\beta = 0 \quad (8)$$

$$A_o^2 \tau^2 = \frac{2|D|}{\delta} \quad (9)$$

$$\psi = -\frac{|D|}{\tau^2} = -\frac{\delta}{2} A_o^2 \quad (10)$$

$$g - l = -\frac{1}{\tau^2} \frac{g}{\Omega_g^2} = -\frac{\gamma}{2} A_o^2 \quad (11)$$

Substitution of these relations into (3) gives

$$a_o(t, T) = A_o \operatorname{sech} \left( \frac{t - t_o}{\tau} \right) \exp \left( -j \frac{\delta}{2} A_o^2 \frac{T}{T_R} + j\theta \right) \quad (12)$$

$$\tau = \frac{1}{A_o} \sqrt{\frac{2|D|}{\delta}}. \quad (13)$$

### III. THE LINEARIZED PERTURBATION EQUATION

Noise is, of course, handled by perturbation theory. If the noise is small compared with the signal, the equations can be linearized. In the adiabatic case, the pulse parameters are changed by the noise, resulting in new pulses that obey the steady state equations. The area of the pulse (amplitude-width product) stays invariant. An adiabatic perturbation of the pulse changes its amplitude, phase, frequency and timing.

We make the ansatz:

$$a(t, T) = [a_s(t - t_o) + \Delta a(t - t_o, T)] \exp \left( -j \frac{\delta}{2} A_o^2 \frac{T}{T_R} \right) \quad (14)$$

where

$$a_s(t) = A_o \operatorname{sech} \left( \frac{t}{\tau} \right). \quad (15)$$

Substitution of the expansion (14) into (1) and disregard of terms higher than first-order in  $\Delta a$  leads to:

$$\begin{aligned} T_R \frac{\partial}{\partial T} \Delta a(t - t_o, T) &= (\mu - j) \left\{ -\frac{\delta}{2} A_o^2 + |D| \frac{\partial^2}{\partial t^2} \right. \\ &\quad \left. + 2\delta a_s^2(t - t_o) \right\} \Delta a + (\mu - j) \delta a_s^2(t - t_o) \Delta a^* \\ &\quad - g_s \left( 1 - \frac{1}{\Omega_g} \frac{\partial}{\partial t} \right) \frac{a_s(t)}{2\tau A_o^2} \int dt a_s(t) (\Delta a + \Delta a^*) \\ &\quad + T_R S(t, T). \end{aligned} \quad (16)$$

We suppressed the indication of the independent variable  $t - t_o$ . The parameter  $g_s$  has been defined as:

$$g_s = \frac{g_o}{P_s T_R} \frac{2\tau A_o^2}{\left\{ 1 + \frac{2\tau A_o^2}{P_s T_R} \right\}^2}. \quad (17)$$

It is convenient to introduce the pulse energy  $w_o$  as a new parameter:

$$w_o = \int dt a_s^2(t) = 2A_o^2 \tau \Rightarrow A_o = \sqrt{\frac{\delta}{2|D|}} \frac{w_o}{2}. \quad (18)$$

We write (16) in the formal way:

$$T_R \frac{\partial}{\partial T} \Delta a(t - t_o) = \hat{A}_t(\Delta a) + T_R S(t - t_o, T) \quad (19)$$

where the operator  $\hat{A}_t$  has been defined:

$$\begin{aligned} \hat{A}_t(\Delta a) &= (\mu - j) \left[ -\frac{\delta}{2} A_o^2 + |D| \frac{\partial^2}{\partial t^2} \right. \\ &\quad \left. + 2\delta a_s^2(t - t_o) \right] \Delta a(t - t_o) \\ &\quad + (\mu - j) \delta a_s^2(t - t_o) \Delta a^*(t - t_o) \\ &\quad - \frac{g_s}{2\tau A_o^2} \left( 1 - \frac{1}{\Omega_g} \frac{\partial}{\partial t} \right) a_s(t - t_o) \\ &\quad \cdot \int dt a_s(\Delta a + \Delta a^*). \end{aligned} \quad (20)$$

In the next section, we express  $\Delta a$  in terms of the perturbations of the four pulse parameters.

#### IV. APPROXIMATE SOLUTION OF PERTURBATION EQUATION

Small changes of pulse energy, phase, timing, and frequency ( $-p = \omega - \omega_o$ , the deviation from the angular carrier frequency [11]) are derivatives of the steady state solution (12):

$$f_w(t) = \frac{\partial a_o}{\partial w_o} = \frac{1}{w_o} \left[ 1 - \frac{t}{\tau} \tanh \left( \frac{t}{\tau} \right) \right] a_s(t) \quad (21)$$

$$f_\theta(t) = \frac{\partial a_o}{\partial \theta_o} = j a_s(t) \quad (22)$$

$$f_t(t) = \frac{\partial a_o}{\partial t_o} = \frac{1}{\tau} \tanh \left( \frac{t}{\tau} \right) a_s(t) \quad (23)$$

$$f_p \equiv j \frac{2}{w_o} t a_s(t). \quad (24)$$

We expand the perturbation in terms of these four functions. The remainder is the continuum that is not part of the pulse.

$$\begin{aligned} \Delta a &= f_w(t - t_o) \Delta w(T) + f_\theta(t - t_o) \Delta \theta(T) \\ &\quad + f_p(t - t_o) \Delta p(T) \\ &\quad + f_t(t - t_o) \Delta t(T) + \Delta a_c(t - t_o, T). \end{aligned} \quad (25)$$

Soliton perturbation theory starts with the above expansion and follows the four perturbation amplitudes  $\Delta w$ ,  $\Delta \theta$ ,  $\Delta p$ , and  $\Delta t$  as they evolve in  $T$  [11] according to the nonlinear Schrödinger equation (NLSE). In this case,  $\Delta w$  and  $\Delta p$  are invariants,  $\Delta \theta$  changes due to the Kerr phase shift change caused by  $\Delta w$ , and  $\Delta t$  changes because the group velocity changes with a change of frequency ( $\Delta p$ ). The master equation (1) contains two additional effects, namely gain dispersion (GD) and APM action. As a consequence of these effects,  $\Delta w$  and  $\Delta p$  do not remain invariant. If these effects are small, i.e., if the parameters  $\mu$  defined in (7) is much less than unity, GD and APM can be treated as perturbations on the NLSE. The coupling to the continuum of the changes of  $\Delta w$  and  $\Delta p$  is of order  $\mu$ , the coupling of the continuum back to the pulse is of order  $\mu$  square. Hence one may use soliton perturbation theory to evaluate the four pulse perturbation parameters to first order in  $\mu$ .

Soliton perturbation theory associates a set of orthogonal adjoint functions with the four functions and the continuum [11]. We choose them so that the products integrate to unity, unless they are orthogonal and the integrals are zero.

$$f_w(t) = 2a_s(t) \quad (26)$$

$$f_\theta(t) = 2j \frac{1}{w_o} \left[ 1 - \frac{t}{\tau} \tanh \left( \frac{t}{\tau} \right) \right] a_s(t) \quad (27)$$

$$f_t(t) = \frac{2}{w_o} t a_s(t) \quad (28)$$

$$f_p(t) = j \left( \frac{2}{w_o \tau} \tanh \frac{t}{\tau} \right) a_s(t). \quad (29)$$

One may use the orthogonality property:

$$\text{Re} \int dt f_i^*(t) f_j(t) = \delta_{ij} \quad (30)$$

to project out the coefficients of the perturbation expansion:

$$\Delta w(T) = \frac{1}{2} \int dt [f_w^*(t) \Delta a(t) + f_w(t) \Delta a^*(t)] \quad (31)$$

$$\Delta \theta(T) = \frac{1}{2} \int dt [f_\theta^*(t) \Delta a(t) + f_\theta(t) \Delta a^*(t)] \quad (32)$$

$$\Delta p(T) = \frac{1}{2} \int dt [f_p^*(t) \Delta a(t) + f_p(t) \Delta a^*(t)] \quad (33)$$

$$\Delta t(T) = \frac{1}{2} \int dt [f_t^*(t) \Delta a(t) + f_t(t) \Delta a^*(t)]. \quad (34)$$

With the aid of these projection operations, one may derive equations of motion of the pulse perturbation parameters.

By making use of the identity

$$\left[ -\frac{\delta}{2} A_o^2 + D \frac{\partial^2}{\partial t^2} + \delta a_s^2 \right] a_s = 0 \quad (35)$$

the following identities can be proven for the operator  $\hat{A}_i$  defined in (20)

$$\hat{A}_i[f_w(t)] = \left[ (\mu - j) \delta A_o^2 - g_s \left( 1 - \frac{1}{\Omega_g} \frac{\partial}{\partial t} \right) \right] \frac{a_s(t)}{w_o} \quad (36)$$

$$\hat{A}_i[f_\theta(t)] = 0 \quad (37)$$

$$\hat{A}_i[f_p(t)] = 2|D|(1 + j\mu)f_i \quad (38)$$

$$\hat{A}_i[f_t(t)] = 0. \quad (39)$$

Substituting (25) into (19) and using the above identities, one obtains the equations of motion for the coefficients of the perturbation expansion:

$$T_R \frac{\partial}{\partial T} \Delta w = [-2g_s + 2\gamma A_o^2] \Delta w + T_R S_w(T) \quad (40)$$

$$T_R \frac{\partial}{\partial T} \Delta \theta = -\delta A_o^2 \frac{\Delta w}{w_o} + T_R S_\theta(T) \quad (41)$$

$$T_R \frac{\partial}{\partial T} \Delta p = -\frac{4}{3} \frac{g}{\Omega_g^2 \tau^2} \Delta p + T_R S_p(T) \quad (42)$$

$$T_R \frac{\partial}{\partial T} = -2|D| \Delta p - \frac{g}{\Omega_g} \frac{\Delta w}{w_o} + T_R S_t(T). \quad (43)$$

The noise sources are defined by

$$S_j(T) = \frac{1}{2} \int dt [f_j^*(t) S(t, T) + f_j(t) S^*(t, T)] \quad (44)$$

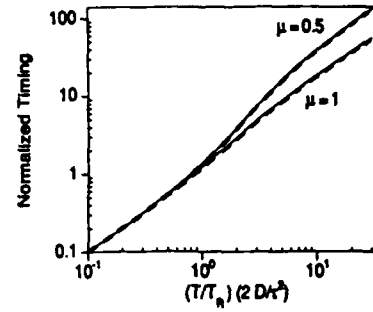


Fig. 1. Timing fluctuations as computed from our theory and by a (numerical) integration that includes back coupling of continuum; the timing is normalized to the noise source. Solid line the analytic result, dashed line computer result;  $g_s = 0$ .

From the preceding equations one draws the following conclusions: The energy will grow exponentially, unless the gain saturation is sufficient to suppress the APM action, as indicated by (40). An energy change couples to the phase evolution, because the change affects the Kerr phase shift, according to (41). Frequency deviations damp out, because the gain of a pulse whose center frequency is off the gain line center is nonuniform across the spectrum and pushes the spectrum back to line center. Finally, a change of frequency and a change of energy both cause a change of group velocity, the first because of GVD, the second since by the Kramers-Kronig relation the gain profile is associated with an index profile that changes along with the gain.

As mentioned earlier, the coupling back from the continuum via gain dispersion and APM or KLM has been neglected causing an error of order  $\mu^2$ . Fig. 1 shows a comparison of the prediction of our theory with the exact numerical solution of the perturbation equation (16) for assumed independent noise sources  $S_j$ . The numerical method of solution has been discussed elsewhere [15]. The ordinate is, of course, proportional to the intensity of the noise source and is normalized to it. It should be noted that the agreement is indeed excellent.

## V. THE NOISE SOURCES

We consider separately the noise introduced by the fluctuations of particular physical quantities.

### Gain Fluctuations

As one can see from the master equation (1), a (long term) fluctuation  $\Delta g(T)$  of the gain introduces an equivalent noise source:

$$S_{\Delta g}(t, T) = \frac{\Delta g(T)}{T_R} \left( 1 - \frac{1}{\Omega_g} \frac{\partial}{\partial t} \right) a_s(t, T). \quad (45)$$

When the laser frequency depends on the gain, which is the case, for example, when different spectral shapes of the gain and the loss force the laser to operate away from the maximum of the gain curve  $\omega_{on}$  gain fluctuations give

also the frequency pulling term (see Fig. 2):

$$S_{\Delta_R}(t, T) = j \frac{\Delta g(T)}{4T_R(\omega_o - \omega_{oo})} \frac{d}{dt} a_s(t, T). \quad (46)$$

This term does not follow directly from (1), but is obtained by extension of the gain operator to carrier frequencies  $\omega_o$  off line center.

### Length and Refractive Index Fluctuations

Fluctuations of the cavity length give rise to two noise contributions:

$$S_{\Delta L}(t, T) = -\frac{\Delta L(T)}{T_R} \left( j \frac{\omega_o}{c} n + \frac{1}{v_g} \frac{d}{dt} \right) a_s(t, T) \quad (47)$$

where  $n$  is the effective index assigned to the total length  $L$ .

The first term in parentheses is the frequency pulling effect of a cavity of varying length. The second term gives the timing jitter caused by a cavity of varying length. Cavity geometry fluctuations in a KLM system may also cause fluctuations of the effective saturable absorber parameter  $\gamma$ , if the cavity is so laid out as to enhance it by operating near the stability boundary of the cavity mode. If  $L$  is again a typical cavity dimension,  $\Delta L \partial \gamma / \partial L$  is an effective noise term in the master equation.

Fluctuations of the refractive index produce noise terms that are analogous to those due to cavity length fluctuations, with  $\Delta L$  replaced by  $L(\Delta n/n)$ .

### Spontaneous Emission Fluctuations

Beside the classical noise sources considered above, spontaneous emission fluctuations are always present into the cavity for reason of quantum mechanical consistency. In a semiclassical approach, the noise introduced by spontaneous emission in a laser is taken into account introducing a white noise source with correlation [16], [17]

$$\langle S_{qn}(T, t) S_{qn}^*(T', t') \rangle = \theta \frac{2g}{T_R} h\nu \delta(T - T') \delta(t - t') \quad (48)$$

where  $\theta$  is the enhancement factor due to the incomplete inversion of the medium.

### Noise Sources in the Equations of Motion

By projecting via the adjoint set of functions, we get

$$S_w(T) = 2 \frac{\Delta g(T)}{T_R} w_o + S_{w, rn}(T) \quad (49)$$

$$S_\theta(T) = \frac{\omega_o}{v_g T_R} \left[ \Delta L(T) + L \frac{\Delta n(T)}{n} \right] + S_{\theta, qn}(T) \quad (50)$$

$$S_p(T) = \frac{\Delta g(T)}{3T_R(\omega_o - \omega_{oo})\tau^2} + S_{p, qn}(T) \quad (51)$$

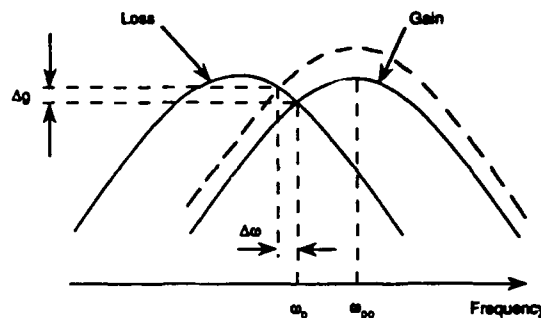


Fig. 2. Gain and loss profiles with different center frequencies for which gain fluctuations cause frequency fluctuations.

$$S_i(T) = \frac{\Delta g(T)}{T_R \Omega_g} - \frac{1}{T_R v_g} \left[ \Delta L(T) + L \frac{\Delta n(T)}{n} \right] + S_{i, qn}(T) \quad (52)$$

where the terms  $S_{i, qn}(T)$ , due to the quantum noise, are independent white noise terms

$$\langle S_{i, qn}(T) S_{h, qn}(T') \rangle = D_{i, qn} \delta_{i, h} \delta(T - T') \quad (53)$$

with diffusion constants

$$D_{w, qn} = 4w_o \theta \frac{2g}{T_R} h\nu \quad (54)$$

$$D_{\theta, qn} = \frac{4}{3w_o} \left( 1 + \frac{\pi^2}{12} \right) \theta \frac{2g}{T_R} h\nu \quad (55)$$

$$D_{p, qn} = \frac{2}{w_o} \frac{4}{3w_o \tau^2} \theta \frac{2g}{T_R} h\nu \quad (56)$$

$$D_{i, qn} = \frac{\pi^2 \tau^2}{3w_o} \theta \frac{2g}{T_R} h\nu. \quad (57)$$

For white noise sources it is convenient to write the correlation functions as

$$\langle S_i(T) S_j(T') \rangle = D_{i, j} \delta(T - T'). \quad (58)$$

The diffusion constants  $D_{i, j}$  form a positive semidefinite matrix. The assumption of white noise simplifies the analytic treatment. Anyway, one should be aware of the fact that, if quantum noise is white, the classical noise sources in general are not. The validity of the results found assuming classical white noise is then restricted to the frequency region inside the actual bandwidth of the noise sources. In the ensuing analysis, the diffusion coefficients will be assumed as given, and not necessarily as those of quantum nature (54), (57).

## VI. THE NOISE SPECTRA AND THE CORRELATION FUNCTIONS

The equations of motion (40)–(43) are of two types. One type is a relaxation equation driven by a noise source, the other has no relaxation time, and is driven by a noise source as well as one of the perturbation variables. In the former case, if the relaxation times are relatively long compared with the inverse bandwidth of the noise sources,

the spectrum of the noise sources can be treated as white and the resulting fluctuations have a Lorentzian spectrum.

Let us define the Fourier transform pair as

$$f(\Omega) = \frac{1}{\sqrt{T_o}} \int dT e^{-j\Omega T} f(T);$$

$$f(T) = \sqrt{T_o} \int \frac{d\Omega}{2\pi} e^{j\Omega T} f(\Omega) \quad (59)$$

where the infinitely long normalization time  $T_o$  has been introduced to avoid the divergence of the fluctuation spectra which is present with the usual definition of Fourier transform. This divergence arises from the fact that delta correlation in time domain corresponds to delta correlation in frequency domain. The inverse of the time  $T_o$  gain the "width" of the delta functions in the frequency domain, and renormalizes their amplitude.

Let us define the two relaxation times:

$$\frac{1}{\tau_w} = (2g_s - 2\gamma A_o^2) \frac{1}{T_R} \quad (60)$$

$$\frac{1}{\tau_p} = \frac{4}{3} \frac{g}{T_R \Omega_g^2 \tau^2} \quad (61)$$

The spectra of the energy fluctuations are then

$$\langle |\delta w(\Omega)|^2 \rangle = \frac{\langle |S_p(\Omega)|^2 \rangle}{\left( \Omega^2 + \frac{1}{\tau_w^2} \right)} \quad (62)$$

and of the frequency fluctuations

$$\langle |\Delta p(\Omega)|^2 \rangle = \frac{\langle |S_p(\Omega)|^2 \rangle}{\left( \Omega^2 + \frac{1}{\tau_p^2} \right)} \quad (63)$$

These spectra couple to the phase and timing fluctuations. If the noise source spectra are treated as white, these latter spectra are of the shape:

$$\frac{1}{\left[ \Omega^2 \left( \Omega^2 + \frac{1}{\tau^2} \right) \right]} \quad (64)$$

which can be expanded into their respective singularities. The leading singularity, which gives the asymptotic behavior for large  $T$ , has a spectrum of the form  $1/\Omega^2$ , corresponds to a random walk. Concentrating on this dominant term, one finds that the phase and timing mean square fluctuations are of the form:

$$\langle |\Delta t(T + T_o) - \Delta t(T_o)|^2 \rangle = D_t T \quad (65)$$

with

$$D_t = \langle |S_T(\Omega = 0)|^2 \rangle + \frac{4D^2 \langle |S_p(\Omega = 0)|^2 \rangle}{T_R^2 / \tau_p^2} \quad (66)$$

and

$$\langle |\Delta \theta(T + T_o) - \Delta \theta(T_o)|^2 \rangle = D_\theta T \quad (67)$$

with

$$D_\theta = \langle |S_\theta(\Omega = 0)|^2 \rangle + \frac{(\delta A_o^2)^2 \langle |S_w(\Omega = 0)|^2 \rangle}{w_o^2 T_R^2 / \tau_w^2} \quad (68)$$

where the coefficients multiplying  $T$  play the role of diffusion coefficients.

The complete expressions for the correlation functions of timing and phase fluctuations is more involved. They are anyway easily calculated from the fluctuation spectra.

The complete expression for the timing jitter spectrum is

$$\langle |\Delta t(\Omega)|^2 \rangle = \frac{4D^2}{T_R^2} \frac{\langle |S_p(\Omega)|^2 \rangle}{\left[ \Omega^2 \left( \Omega^2 + \frac{1}{\tau_p^2} \right) \right]} + \frac{\langle |S_t(\Omega)|^2 \rangle}{\Omega^2} \quad (69)$$

and for the phase noise spectrum

$$\langle |\Delta \theta(\Omega)|^2 \rangle = \frac{(\delta A_o^2)^2}{T_R^2} \frac{\langle |S_w(\Omega)|^2 \rangle}{\left[ \Omega^2 \left( \Omega^2 + \frac{1}{\tau_w^2} \right) \right]} + \frac{\langle |S_\theta(\Omega)|^2 \rangle}{\Omega^2} \quad (70)$$

The timing mean square fluctuations can be easily obtained by using the relationship

$$\begin{aligned} \langle |\Delta t(T + T_o) - \Delta t(T_o)|^2 \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle |\Delta t(\Omega) (e^{j\Omega T} - 1)|^2 \rangle d\Omega \\ &= \frac{2}{\pi} \int_0^{\infty} \langle |\Delta t(\Omega)|^2 \rangle [1 - \cos(\Omega T)] d\Omega. \end{aligned} \quad (71)$$

Assuming white noise spectra for the frequency and timing noise sources, we get

$$\begin{aligned} \langle |\Delta t(T + T_o) - \Delta t(T_o)|^2 \rangle &= \frac{4D^2}{T_R^2} D_{p,p} \tau_p^3 \left( \frac{T}{\tau_p} - 1 + e^{-T/\tau_p} \right) + D_{t,t} T. \end{aligned} \quad (72)$$

It is interesting to calculate the two asymptotic limits of (72). Neglecting the contribution of  $D_{t,t}$ , for  $T/\tau_p \ll 1$  we have

$$\langle |\Delta t(T + T_o) - \Delta t(T_o)|^2 \rangle \approx \frac{4D^2}{T_R^2} \frac{D_{p,p} \tau_p T^2}{2} \quad (73)$$

while for  $T/\tau_p \gg 1$

$$\langle |\Delta t(T + T_o) - \Delta t(T_o)|^2 \rangle \approx \frac{4D^2}{T_R^2} D_{p,p} \tau_p^2 T \quad (74)$$

The variance of the timing fluctuations is unbounded, as expected from the absence of any time reference in passively modelocked lasers. It grows for small times quadratically and for large times linearly with  $T$ . The quadratic growth with time arises from the  $1/\Omega^4$  dependence for  $\Omega \gg 1/\tau_p$ , while the linear dependence corresponds to the  $1/\Omega^2$  spectrum for  $\Omega \ll 1/\tau_p$ .

It is worth noting that for negligible initial frequency fluctuations a direct integration in the time domain of (42) and (43) gives, in the asymptotic limit  $T \ll \tau_p$

$$\langle |\Delta t(T + T_0) - \Delta t(T_0)|^2 \rangle \approx \frac{4D^2}{T_R^2} D_{p,p} T^3, \quad (75)$$

different from the previous result. This  $T^3$  dependence is closely related to the cubic dependence of timing fluctuations on propagation distance in long haul soliton communication systems known as the Gordon-Haus effect [16]. The different asymptotic behavior of (73) is due to the different natures of the two processes: in a laser we are dealing with steady-state properties, that do not exist in the propagation of a soliton in a fiber, by its nature a transient process.

The phase mean square fluctuations can be evaluated in a similar way, obtaining

$$\begin{aligned} \langle |\Delta \theta(T + T_0) - \Delta \theta(T_0)|^2 \rangle \\ = \frac{(2\delta A_0^2)^2}{w_0^2 T_R^2} D_{w,w} \tau_w^3 \left( \frac{T}{\tau_w} - 1 + e^{-T/\tau_w} \right) + D_{\theta,\theta} T. \end{aligned} \quad (76)$$

Other spectra and correlation functions are of interest. We will assume for the sake of simplicity that the noise sources are white. From the spectrum of energy fluctuations, we get the correlation function of the energy fluctuations

$$\begin{aligned} \langle w(T + T_0)w(T_0) \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\Omega T} \langle |w(\Omega)|^2 \rangle d\Omega \\ &= \frac{\tau_w}{2} D_{w,w} e^{-|T|/\tau_w}. \end{aligned} \quad (77)$$

The cross-correlation spectrum of time and frequency fluctuations reads

$$\langle \Delta t^*(\Omega) \Delta w(\Omega) \rangle = \frac{-2|D|D_{p,w}}{-j\Omega(-j\Omega + 1/\tau_p)(j\Omega + 1/\tau_w)} \quad (78)$$

and the cross-correlation function

$$\begin{aligned} \langle [\Delta t(T + T_0) - \Delta t(T_0)] \Delta w(T_0) \rangle \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{-j\Omega T} - 1) \langle \Delta t^*(\Omega) \Delta w(\Omega) \rangle d\Omega \\ = -2 \operatorname{sign}(T) |D| D_{p,w} \frac{\tau_p \tau_w}{\tau_p + \tau_w} \tau (1 - e^{-|T|/\tau}) \end{aligned} \quad (79)$$

where  $\tau$  is equal to  $\tau_p$  for  $T > 0$  and to  $\tau_w$  for  $T < 0$ .

## VII. THE MEASUREMENT OF NOISE

The most direct way of measuring the different noise spectra would be to construct a homodyne detector with a pulsed local oscillator having one of the adjoint func-

tions  $f_i$  as the pulse shape. This pulse would be derived from the mode-locked pulse and delayed by the time  $T$ . Successive samples of the square of the detected output would be collected. Thus one would arrive at the mean square fluctuations of the four pulse parameters. In some laser systems, the delays would have to be considerable, since the relaxation times of some solid-state lasers are many microseconds long. Delay fiber "lines" would distort the pulses and the experiment would not be all that simple. An alternative way is based on the detection of the low frequency spectrum of the detected pulse [21]. Let us concentrate on this approach.

The periodic pulse train has the time dependence:

$$a(t, T) = \sum_m A_0 \operatorname{sech} \left( \frac{t - \Delta t - mT_R}{\tau} \right) e^{j(\omega_0 - \Delta p)t} e^{j\Delta \theta}. \quad (80)$$

The field amplitude  $a(t, T)$  is a function of two time variables,  $t$  and  $T$ ; the former is the fast time variable in terms of which the pulse envelope is expressed; the latter is the time variable of the slow time scale in terms of which the noise is expressed. The dependence on  $T$  is implicit in the time dependences of the perturbations of energy,  $\Delta w(T)$ , of frequency,  $\Delta p(T)$ , of phase,  $\Delta \theta(T)$ , and of timing,  $\Delta t(T)$ . The pulse repeats every round-trip time  $T_R$ . Since the laser relaxation rates are slow compared with one round-trip time, the fluctuations produce narrow-band spectra around each of the Fourier components of the periodic pulse train separated by  $\Omega_0 = 2\pi/T_R$ . One may decompose the function  $a(t, T)$  into a sum of Fourier components:

$$a(t, T) = \pi \frac{A_0 \tau}{T_R} \sum_n \operatorname{sech} \left[ \frac{\pi}{2} n \Omega_0 \right] e^{j(n\Omega_0 + \omega_0 - \Delta p)t} e^{-jn\Omega_0 \Delta t} e^{j\Delta \theta}. \quad (81)$$

When the pulse train impinges on a detector, the  $k$ th Fourier component of the detector current is produced by the beat of the  $n$ th Fourier component of the amplitude with the  $(n + k)$ th Fourier component:

$$\begin{aligned} [i(t, T)]_k &= \eta \frac{e}{\hbar \omega_0} \left( \pi \frac{A_0 \tau}{T_R} \right)^2 \sum_n \operatorname{sech} \left( \frac{\pi \tau}{2} m \Omega_0 \right) \\ &\quad \cdot \operatorname{sech} \frac{\pi \tau}{2} [(m + k)\Omega_0] e^{jk\Omega_0 \Delta t} e^{-jk\Omega_0 \Delta t} \end{aligned} \quad (82)$$

where  $\eta$  is the quantum efficiency of the detector and  $e$  is the electron charge. We note that the phase fluctuations  $\Delta \theta$  and the frequency shift  $\Delta p$  do not appear in the Fourier components of the current. In order to measure them, one needs a different experiment. We shall return to the discussion of such a measurement further on.

Since the pulsetrain contains very many Fourier components (1000 or more), one may expand the expressions to first order in  $k\tau\Omega_0$ . Further, one notes that the pulse-width itself is subject to fluctuations, along with the amplitude  $A_0$ . If the pulse is soliton like, its area is invariant

and

$$\Delta(w\tau) = \Delta(2A_o^2\tau^2) = 0 \quad (83)$$

so that

$$\frac{\Delta w}{w} = -\frac{\Delta\tau}{\tau} \quad (84)$$

The amplitude of the  $k$ th Fourier component of the current becomes:

$$i_k(T) = \eta \frac{e}{\hbar\omega_o} \left( \pi \frac{A_o\tau}{T_R} \right)^2 \sum_m \operatorname{sech}^2 \left( \frac{\pi}{2} \tau m \Omega_o \right) \cdot \left[ 1 - \frac{\pi}{2} \tau k \Omega_o \tanh \left( \frac{\pi}{2} \tau m \Omega_o \right) \right] \cdot \left[ 1 - 2 \frac{\Delta\tau}{\tau} \left( \frac{\pi}{2} \tau m \Omega_o \right) \tanh \left( \frac{\pi}{2} \tau m \Omega_o \right) - \frac{\Delta\tau}{\tau} \left( \frac{\pi}{2} \tau k \Omega_o \right) \tanh \left( \frac{\pi}{2} \tau m \Omega_o \right) \right] e^{-jk\Omega_o\Delta t}. \quad (85)$$

One may replace the sum over  $m$  by an integral. Only the symmetric functions of  $m$  contribute. The result is

$$i_k(T) = \mathfrak{N} \left[ 1 + f_k \frac{\Delta w}{w} \right] e^{-jk\Omega_o\Delta t} \quad (86)$$

where

$$\mathfrak{N} = \eta \frac{e}{\hbar\omega_o} \left( \pi \frac{A_o\tau}{T_R} \right)^2 \frac{4}{\pi\tau\Omega_o} \quad (87)$$

$$f_k = 1 + \frac{1}{3} \left( \frac{\pi}{2} k\tau\Omega_o \right)^2. \quad (88)$$

Consistently with the first-order expansion in  $\pi k\tau\Omega_o/2$  we will neglect the second term in the definition of  $f_k$  by setting  $f_k = 1$ . If the energy and timing fluctuations are independent, the correlation function of the current is the sum of the correlation functions of  $\exp(-jk\Omega_o\Delta t)$  and of  $\Delta t$ , and so is the spectrum. If the timing fluctuations go through a random walk, the spectrum of  $\exp(-jk\Omega_o\Delta t)$  is Lorentzian. The spectrum of  $\Delta\tau$  is added to it. The power of the Fourier components of the former increases with  $k^2$ , the latter has a constant contribution plus one that increases with  $k^2$  and another that increases with  $k^4$ . Thus the two fluctuations  $\Delta w(T)$  (or  $\Delta\tau(T)$ ) and  $\Delta t(T)$  contribute to the current spectrum. They are independently measurable.  $\Delta t(T)$  experiences a random walk with the diffusion constant  $D_t$ . Hence it produces a linewidth contribution to the  $k$ th Fourier component that is

$$\Delta\Omega = k^2 D_t / 2 \quad (89)$$

The latter has a contribution that is independent of  $k$ . Thus the two fluctuations  $\Delta w(T)$  (or  $\Delta\tau(T)$ ) and  $\Delta t(T)$  contribute to the current spectrum. They are independently measurable on the basis of their different  $k$  dependence.

When  $\Delta t$  and  $\Delta\tau$  are correlated, and/or the timing fluctuations are not those of a simple random walk, the spec-

trum is more complicated. Let us define the spectrum of the amplitude of  $k$ th Fourier component of the current fluctuations as

$$i_k(\Omega) = \int_{-\infty}^{\infty} e^{-j\Omega T} C_k(T) dT \quad (90)$$

where

$$C_k(T) = \langle i_k(T + T_o) i_k^*(T_o) \rangle. \quad (91)$$

Since  $C_k(T) = C_k^*(-T)$ , the spectrum can also be written as

$$i_k(\Omega) = 2 \operatorname{Re} \int_0^{\infty} e^{-j\Omega T} C_k(T) dT. \quad (92)$$

The correlation function follows from (86)

$$C_k(T) = \mathfrak{N}^2 \left\langle \left\{ 1 + \frac{1}{w} [\Delta w(T + T_o) + \Delta w^*(T_o)] + \frac{1}{w^2} [\Delta w(T + T_o) \Delta w^*(T_o)] \right\} \cdot e^{-jk\Omega_o[\Delta t(T + T_o) - \Delta t(T_o)]} \right\rangle. \quad (93)$$

For Gaussian processes the average can be evaluated [20] and the result is

$$C_k(T) = \mathfrak{N}^2 \left\{ 1 + \frac{1}{w^2} \langle \Delta w(T + T_o) \Delta w(T_o) \rangle - \frac{k^2 \Omega_o^2}{w^2} \langle [\Delta t(T + T_o) - \Delta t(T_o)] \Delta w(T) \rangle \cdot \langle [\Delta t(T + T_o) - \Delta t(T_o)] \Delta w(T_o) \rangle - j \frac{k\Omega_o}{w} [\langle [\Delta t(T + T_o) - \Delta t(T_o)] \Delta w(T) \rangle + \langle [\Delta t(T + T_o) - \Delta t(T_o)] \Delta w(T_o) \rangle] \right\} \cdot \exp - \frac{1}{2} k^2 \Omega_o^2 \langle [\Delta t(T + T_o) - \Delta t(T_o)]^2 \rangle. \quad (94)$$

Substituting in this expression the values of the correlation functions, and taking into account that

$$\begin{aligned} & \langle [\Delta t(T + T_o) - \Delta t(T_o)] \Delta w(T) \rangle \\ &= -\langle [\Delta t(-T + T_o) - \Delta t(T_o)] \Delta w(T_o) \rangle \end{aligned} \quad (95)$$

we get, for  $T > 0$  and for white noise sources

$$C_k(T) = \mathfrak{N}^2 \left\{ 1 + a_1 e^{-(T/\tau_w)} - a_2^2 k^2 \tau_p \tau_w (1 - e^{-(T/\tau_w)}) \cdot (1 - e^{-(T/\tau_p)}) + k a_2 [\tau_w (1 - e^{-(T/\tau_w)}) + \tau_p (1 - e^{-(T/\tau_p)})] \right\} \cdot \exp \left[ -k^2 a_3 \left( \frac{T}{\tau_p} - 1 + e^{-(T/\tau_p)} \right) \right] \quad (96)$$

where

$$c_1 = \frac{1}{w^2} \frac{\tau_w}{2} D_{w,w} \quad (97)$$

$$c_2 = 2|D| \frac{1}{wT_R} \Omega_o \frac{\tau_p \tau_w}{\tau_p + \tau_w} D_{w,p} \quad (98)$$

$$c_3 = \frac{2D^2}{T_R^2} D_{p,p} \Omega_o^2 \tau_p^3. \quad (99)$$

Since

$$D_{w,w} D_{p,p} \geq D_{w,p}^2 \quad (100)$$

for the positive semidefiniteness of the correlation matrix, the coefficients  $c_i$  have to meet the condition

$$c_1 c_3 \geq \frac{\tau_p}{4\tau_w} (\tau_p + \tau_w)^2 c_2^2 \quad (101)$$

where the equality sign holds for perfectly correlated noise sources, when energy and frequency fluctuations are driven by the same noise sources (e.g., gain fluctuations).

The spectrum can be easily evaluated analytically, since the only integral which is involved can be easily performed with the substitution  $e^{-x} \rightarrow y$ ,

$$\int_0^\infty \exp(-ax - be^{-x}) dx = b^{-a} \gamma(a, b) \quad (102)$$

where

$$\gamma(a, b) = \int_0^b e^{-t} t^{a-1} dt \quad (103)$$

is the incomplete gamma function. If we define

$$\phi(\alpha, \Omega) = \tau_p e^{k^2 c_3 (k^2 a_3)^{(j\Omega\tau_p - \alpha - k^2 c_3)}} \cdot \gamma[-(j\Omega\tau_p - \alpha - k^2 c_3), k^2 c_3] \quad (104)$$

the spectrum of the  $k$ th Fourier component becomes

$$\begin{aligned} i_k(\Omega) = 2\Re \left\{ \phi(0, \Omega) + c_1 \phi\left(\frac{\tau_p}{\tau_w}, \Omega\right) \right. \\ \left. - c_2^2 k^2 \tau_p \tau_w \left[ \phi(0, \Omega) - \phi\left(\frac{\tau_p}{\tau_w}, \Omega\right) - \phi(1, \Omega) \right. \right. \\ \left. \left. + \phi\left(\frac{\tau_p}{\tau_w} + 1, \Omega\right) \right] \right. \\ \left. + jkc_2 \left[ (\tau_p + \tau_w)\phi(0, \Omega) - \tau_p \phi(1, \Omega) \right. \right. \\ \left. \left. - \tau_w \phi\left(\frac{\tau_p}{\tau_w}, \Omega\right) \right] \right\}. \quad (105) \end{aligned}$$

This expression has been used to plot the curves shown in Fig. 3. When intensity and timing fluctuations are uncorrelated, for  $c_3 = 0$ , the spectrum is symmetric. Asymmetry appears when  $c_3 \neq 0$  when intensity and phase fluctuations are correlated, as shown in Fig. 4.

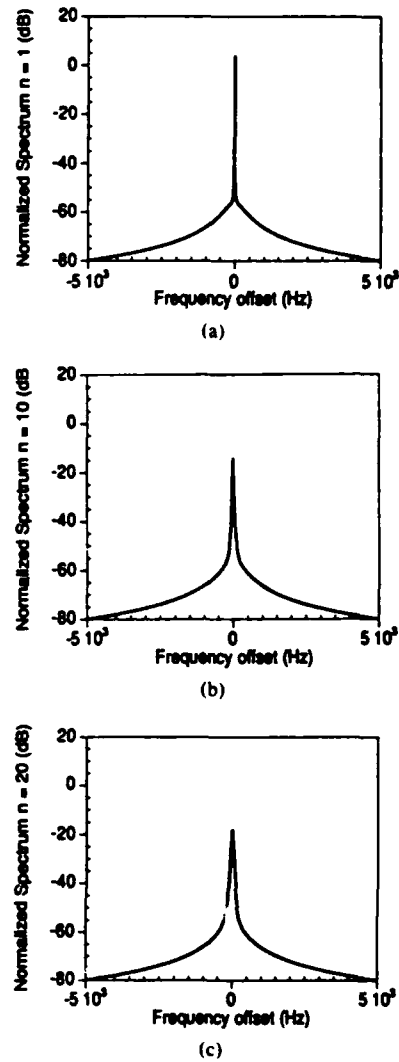


Fig. 3. The spectra of different current Fourier components for uncorrelated intensity and timing fluctuations. The values of the parameters are:  $\tau_w = 0.5$  ms,  $\tau_p = 8.4$  ms,  $c_1 = 25 \times 10^{-4}$  (5% intensity fluctuations),  $c_2 = 0$ ,  $c_3 = 0.0073$ .

It is interesting to give a simplified expression of  $i_k(\Omega)$  for large  $\Omega$ . Large  $\Omega$  corresponds to small  $T$ , for which the exponential in (86) can be expanded to first order obtaining

$$i_k(T) \approx \Re \left[ 1 + \frac{1}{w} \Delta w(T) \right] \left[ 1 - jk\Omega_o \Delta t(T) \right]. \quad (106)$$

From this expression the spectrum is directly evaluated

$$\begin{aligned} \langle |i_k(\Omega)|^2 \rangle \approx \Re \left\{ 2\pi\delta(\Omega) + \frac{1}{w^2} \langle |\Delta w(\Omega)|^2 \rangle \right. \\ \left. + k^2 \Omega^2 \langle |\Delta t(\Omega)|^2 \rangle \right. \\ \left. + jk\Omega_o [\langle \Delta t^*(\Omega) \Delta w(\Omega) \rangle \right. \\ \left. - \langle \Delta t(\Omega) \Delta w^*(\Omega) \rangle] \right\}. \quad (107) \end{aligned}$$

Von der Linde was the first to propose the use of (107) to measure the energy fluctuations and the timing jitter

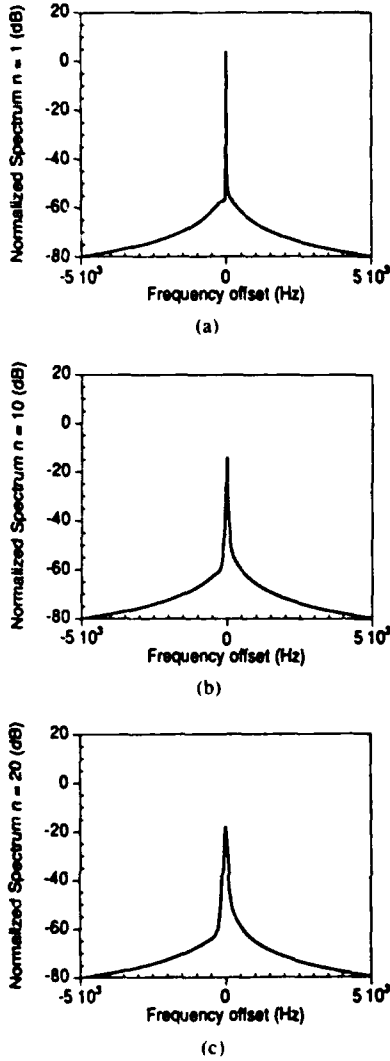


Fig. 4. The spectra of different current Fourier components for correlated intensity and timing fluctuations. The values of the parameters are the same as in Fig. 3, and  $c_2 = \sqrt{4c_1 c_3 \tau_w / [\tau_p (\tau_p + \tau_w)^2]}$  [see (101)].

spectra in mode-locked lasers. He considered the case of uncorrelated energy and timing fluctuations, when the quantity within square brackets in (107) is zero. He noted that the  $k$ th Fourier component of the detected current is a Dirac delta function, corresponding to the noiseless laser, plus the sum of the energy fluctuation spectrum and of  $k^2$  times the timing jitter spectrum. The contributions of energy fluctuations and timing jitter can be resolved by using their different  $k$  dependences. In the original paper, the theory was developed mainly for active modelocking where, due to the active modulation, timing fluctuations are a stationary process. Later on, this theory has also been applied to the study of the time jitter of passive mode-locked systems. Since in passive mode-locking the timing fluctuations are not a stationary process, the von der Linde's method has to be used carefully. To see the differences, it is useful to consider the case in which the timing jitter is a stationary process, by introducing a restoring force in the equation of motion for  $\Delta t(T)$ . This accounts for the introduction of a modulator, locked to an

external oscillator, into the laser cavity as proposed in [22] to achieve self starting and reduction of the timing jitter. Equation (43) is modified by adding the term  $-\Delta t/\tau_t$  to the RHS, where  $\tau_t$  is the decay time of the timing fluctuations. The spectrum of the timing jitter becomes

$$\langle |\Delta t(\Omega)|^2 \rangle = \frac{4D^2}{T_R^2} \frac{\langle |S_p(\Omega)|^2 \rangle}{\left[ \left( \Omega^2 + \frac{1}{\tau_i^2} \right) \left( \Omega^2 + \frac{1}{\tau_p^2} \right) \right]} + \frac{\langle |S_t(\Omega)|^2 \rangle}{\Omega^2 + \frac{1}{\tau_t^2}} \quad (108)$$

The  $1/\Omega^2$  divergence typical of random walk present in (69) is removed as expected because timing jitter is now a stationary process. The timing jitter correlation function becomes

$$\begin{aligned} \langle |\Delta t(T + T_o) - \Delta t(T_o)|^2 \rangle \\ = \frac{4D^2}{T_R^2} D_{p,p} \frac{\tau_p^2 \tau_t^2}{\tau_i^2 - \tau_p^2} [\tau_t(1 - e^{-|T|/\tau_t}) \\ - \tau_p(1 - e^{-|T|/\tau_p})] + D_{t,t} \tau_t(1 - e^{-|T|/\tau_t}). \end{aligned} \quad (109)$$

The expression for the correlation function of the  $k$ th Fourier component of the photocurrent, in the case of uncorrelated time and energy fluctuations, is

$$\begin{aligned} C_k(T) = \Re^2 \left[ 1 + \frac{1}{w^2} \langle \Delta w(T + T_o) \Delta w(T_o) \rangle \right] \\ \cdot \exp \left[ -\frac{1}{2} k^2 \Omega_o^2 \langle |\Delta t(T + T_o) - \Delta t(T_o)|^2 \rangle \right]. \end{aligned} \quad (110)$$

The correlation function of the energy fluctuations is given by (77). The spectrum of the  $k$ th Fourier component is given by the Fourier transform of  $C_k(T)$ . It is interesting to see what happens for  $|T| \rightarrow \infty$ : the timing fluctuations given by (109) approach asymptotically a constant value. This means that the fluctuation spectrum at  $\Omega = 0$  is divergent. Let us examine the nature of this divergence by considering, for simplicity, the case of  $\tau_t \gg \tau_p$ . In this limit, the  $k$ th component of the photocurrent spectrum becomes

$$\begin{aligned} \langle |i_k(\Omega)|^2 \rangle = \int_{-\infty}^{\infty} e^{-j\Omega T} (1 + d_1 e^{-|T|/\tau_w}) \\ \cdot \exp [-d_3 k^2 \tau_t (1 - e^{-|T|/\tau_t})] dT \end{aligned} \quad (111)$$

where  $d_1$  and  $d_3$  are positive constants proportional to the noise diffusion constants:

$$\begin{aligned} d_1 &= \frac{\tau_w D_{w,w}}{2w_o^2} \\ d_3 &= \frac{2\Omega_o^2 D^2 D_{p,p} \tau_p^2}{T_R^2} \end{aligned}$$

By expanding  $\exp [d_3 k^2 \tau_i e^{-(|T|/\tau_i)}]$  into a series and integrating we get

$$\begin{aligned} \langle |i_k(\Omega)|^2 \rangle = & e^{-d_3 k^2 \tau_i} \left\{ 2\pi\delta(\Omega) + \frac{d_1}{\tau_w} \frac{2}{\Omega^2 + \frac{1}{\tau_w^2}} \right. \\ & + \sum_{n=1}^{\infty} \frac{(d_3 k^2 \tau_i)^n}{n!} \left[ d_1 \frac{\frac{2}{\tau_w} + \frac{2n}{\tau_i}}{\Omega^2 + \left(\frac{1}{\tau_w} + \frac{2n}{\tau_i}\right)^2} \right. \\ & \left. \left. + \frac{\frac{n}{\tau_i}}{\Omega^2 + \frac{n^2}{\tau_i^2}} \right] \right\}. \end{aligned} \quad (112)$$

For  $\Omega = 0$  the series is convergent, and the only term left is the Dirac delta function. We have found that the divergence of the current spectrum at  $\Omega = 0$  is due to a delta function. Equation (107) is obtained by keeping in the exact expression only the terms of first order in  $d_1$  and  $d_3$ . Summarizing, the  $k$ th order of the low frequency noise spectrum of the photocurrent is a delta function at  $\Omega = 0$  plus a broad-band noise spectrum which, at first-order, is the sum of the energy and  $k^2$  times the timing jitter spectra. The first order expansion of the exponential that leads to (107) is then justified, because the terms left over in the expansion are small for all values of  $\Omega$ . In Fig. 5 the result of the approximate and exact expressions are compared. The solid line is the plot of

$$\begin{aligned} i_k(\Omega) = & e^{-d_3 k^2 \tau_i} 2\tau_i \operatorname{Re} \left[ (-d_3 k^2 \tau_i)^{-j\Omega\tau_i} \gamma(j\Omega\tau_i, -d_3 k^2 \tau_i) \right. \\ & + d_1 (-d_3 k^2 \tau_i)^{(-j\Omega\tau_i - (1/\tau_w))} \\ & \left. \cdot \gamma \left( j\Omega\tau_i + \frac{1}{\tau_w}, -d_3 k^2 \tau_i \right) \right] \end{aligned} \quad (113)$$

which is equivalent to the sum of all the terms of (112) and has indeed a delta function singularity in the origin, while the dashed line is the plot of the only first-order terms in  $d_1$  and  $d_3$ . The approximate and exact analyses give indistinguishable results.

The general structure of the photocurrent spectra with a Dirac delta function and a broad-band background is common to all the cases in which the presence of a restoring force in the timing fluctuations makes the timing jitter a stationary process like, e.g., the active mode locking.

In the case of pure passive mode-locking we have a different picture. In this case, the linear divergence of the timing fluctuations for long  $|T|$  due to the absence of any restoring force prevents the Fourier transform of  $C_k(T)$  to diverge for  $\Omega = 0$ . This means that now the Dirac delta

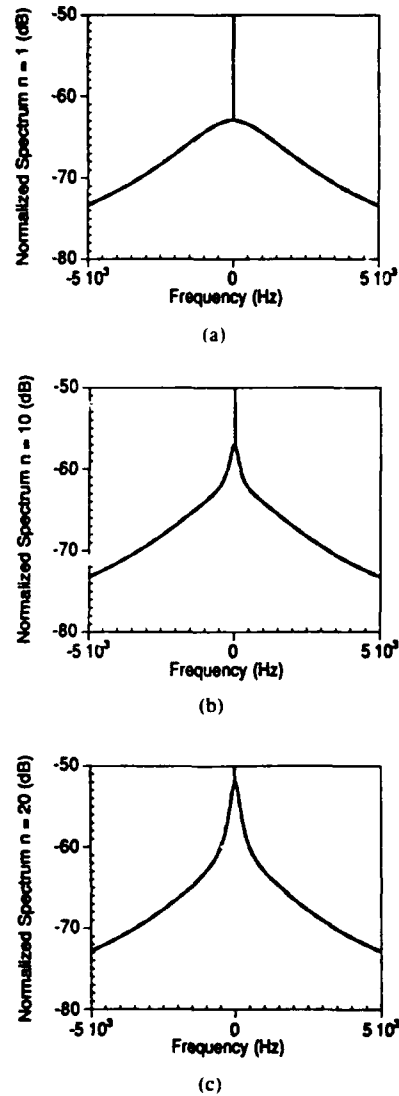


Fig. 5. The spectra of different current Fourier components in presence of an active mode-locking restoring force. The values of the parameters are:  $\tau_w = 0.5$  ms,  $\tau_i = 1$  ms,  $d_1 = 25 \times 10^{-4}$ ,  $d_3 = 0.0073$ .

function at  $\Omega = 0$  of the approximate expression (107) is nonphysical, and it is present only because we have performed a first-order expansion of the exponential for times where this approximation is not allowed. Equation (107) is a good approximation of the actual photocurrent spectrum only down to a cutoff frequency where the variance of the timing fluctuations times  $k^2 \Omega_0^2$  becomes of the order of 1. This means that, in principle, the timing jitter spectrum at very low frequency cannot be directly measured by the method proposed by von der Linde which is based on the validity of the linearized equation (107). The approximate and the exact expressions of the photocurrent spectrum, calculated assuming zero energy fluctuations, are compared in Figs. 6 and 7 assuming different values for the diffusion constant for the frequency. In Fig. 6 the values of the parameters have been chosen to fit the results of [22]. One sees that the spectrum rises proportionally to  $k^2$ , and that the comparison between approximate and exact expressions is indeed excellent down to 100 Hz. Since

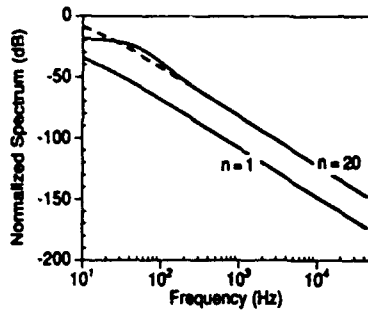


Fig. 6. The spectrum of the current fluctuations of the first and 20th Fourier component of the current. The values of the parameters, chosen to fit the results of [22], are  $\tau_p = 8.4$  ms,  $c_1 = 0$ ,  $c_2 = 0$  (zero intensity fluctuations),  $c_3 = 0.0073$ . Dashed approximate, solid exact.

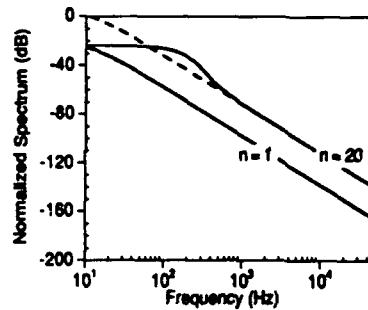


Fig. 7. The spectrum of the current fluctuations of the first and 20th Fourier component for higher values of the frequency diffusion constant. The values of the parameters are:  $\tau_p = 8.4$  ms,  $c_1 = 0$ ,  $c_3 = 0.073$ .

the cutoff frequency below which the approximate expression fails is proportional to  $k^2$ , we see a large deviation from the exact theory only when we consider very high harmonics.

When timing jitter increases, the approximate and exact analyses give different results for low harmonic numbers. This is shown in Fig. 7, where the plots of Fig. 6 are repeated for a ten times larger value of the frequency diffusion constant. In this last case the exact and the approximate analyses give different results even for  $n = 20$ . Consequently, the shape of the timing jitter spectrum (given by the approximate analysis) cannot be obtained by the analysis of the high-order components of the photocurrent spectrum. On the other hand, the lower order components are affected by the intensity noise as shown in Fig. 8 where the case of Fig. 7 has been plotted by adding an intensity noise of 5%.

Phase and frequency fluctuations need a different way of detection. A cascade of Fabry-Perot interferometers can isolate the fundamental  $n = 0$  of the pulse spectrum

$$a_o(t, T) = \pi \frac{A_o \tau}{T_R} e^{j\omega_d t} e^{-j\Delta p t} e^{j\Delta \theta} \quad (114)$$

When the diffusion of the phase can be approximated to a random walk, it causes a Lorentzian shape of the frequency spectrum whose width is equal to the diffusion constant of the phase [23]. When this approximation can-

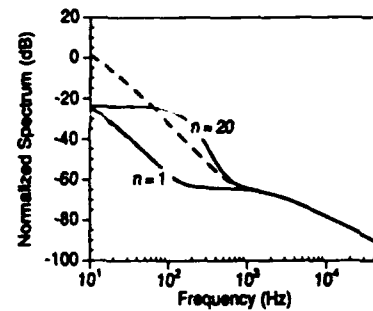


Fig. 8. The spectrum of the current fluctuations for the first and 20th Fourier component for the same values of Fig. 7 and  $\tau_w = 0.5$  ms,  $c_2 = 25 \times 10^{-4}$  (5% intensity fluctuations). Dashed approximate, solid exact.

not be performed, its complete expression is

$$\begin{aligned} S_{\Delta \theta}(\Omega) &= \int_{-\infty}^{\infty} e^{-j\Omega T} e^{-(1/2)\langle (\Delta \theta(T+T_0) - \Delta \theta(T_0))^2 \rangle} dT \\ &= 2 \operatorname{Re} \{ \tau_w e^{b_1} b_1^{(j\Omega \tau_p - b_1)} \gamma[-(j\Omega \tau_w - b_1), b_1] \} \end{aligned} \quad (115)$$

where

$$b_1 = \frac{(\delta A_o^2)^2}{2\omega_o^2 T_R^2} D_{w,w} \tau_w^3. \quad (116)$$

The spectrum of  $e^{-j\Delta p t}$  is Gaussian and its expression is

$$\begin{aligned} S_{\Delta p}(\Omega) &= \int_{-\infty}^{\infty} e^{-j\Omega t} e^{-(1/2)\langle (\Delta p(T)^2)^2 \rangle} dt \\ &= \sqrt{2\pi} e^{-(\Omega^2/2\langle \Delta p(T_0)^2 \rangle)}. \end{aligned} \quad (117)$$

The net spectrum is the convolution of the two spectra.

## VIII. EXPERIMENTAL RESULTS

Let us examine briefly the existing experimental data on timing jitter of passive mode-locked lasers. Many experimental results have been published so far on timing jitter of colliding pulse mode-locked (CPM) lasers [24], [25] and of color-center lasers [26]. Even though our model applies to APM and KLM lasers, all reported experimental configurations involve intracavity negative dispersion to compensate for the self-phase modulation. The mechanism that produces the coupling between frequency fluctuations and timing fluctuations (the dependence on frequency of the group velocity) is expected to be active also in these cases. Generally, it is difficult to obtain, from the published experiments, all the information required on the numerical values for the parameters of the theory to make quantitative comparisons. However, the very specific qualitative predictions of the theory can be compared with available experimental information. In all the reported timing jitter spectra, when no particular arrangement is used to reduce the timing jitter, the power spectra drop 40 dB/decade, which corresponds to the expected  $1/\Omega^4$  behavior. More interesting is the jitter measurement in [22], where the investigated laser closely corresponds to our model. In this case the  $1/\Omega^4$  behavior

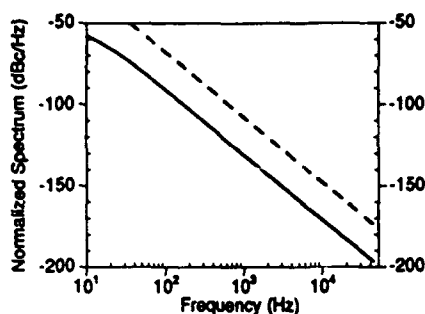


Fig. 9. The timing jitter spectrum of an assumed spontaneous emission excitation (solid line) compared to the experimental results of [22] (dashed line).

is particularly evident. All these experimental results suggest that the source of timing fluctuations is not the direct broadband excitation of timing jitter which would give a  $1/\Omega^2$  contribution, but rather a broadband source of carrier frequency noise that produce timing jitter via the group velocity dependence on the carrier frequency.

Intensity noise and phase noise are usually far from the quantum limit in solid-state and gas lasers. Unavoidable gain fluctuations induced by fluctuations in the power supply and length fluctuations due to environmental perturbations are dominant. On the contrary, as already pointed out, when the laser runs at the maximum of the gain bandwidth there are no evident classical noise sources for frequency fluctuations, which are the dominant contribution to the timing fluctuations at low frequency. For this reason, it is of interest to evaluate the timing jitter spectrum as driven by quantum noise. We have gone through a numerical example using the following parameters:  $2g$  (incremental gain per pass) =  $0.06$ ,  $T_R = 1.16 \cdot 10^{-8}$  s,  $w_o = 2 \cdot 10^{-7}$  J,  $2D = -4000$  fs<sup>2</sup>,  $\lambda = 840$  nm,  $\Omega_g = 1.6 \cdot 10^{15}$  s<sup>-1</sup>,  $\tau = 60/1.763$  fs, noise enhancement factor  $\theta = 1$  (assuming that the lower level of the laser transition is empty). These data have been obtained from [22] for the Ti:Al<sub>2</sub>O<sub>3</sub> laser described there. The results are shown in Fig. 9. Here, the solid line is the spectrum obtained with these data, the dashed line is the spectral power of the timing jitter obtained from [22]. The spectra are multiplied by the normalization factor  $\Omega_o^2$ , to be consistent with the units of dB over the carrier (dBc) usually reported in the experimental literature. The frequency dependence of the experimental results is well reproduced. The peaks at 50, 150 Hz, etc., which show up in the experimental spectrum corresponding to amplitude noise in the pump laser, can be attributed to low frequency fluctuations of the spontaneous emission. The theoretical spectrum obtained assuming only quantum fluctuations as a source of frequency noise is around 20 dB below the experimental spectrum. This corresponds to calculated rms timing jitter one order of magnitude lower than the measured one. The discrepancy may be due to an incomplete knowledge of the experimental parameters, to offset of the gain line center from the passive cavity loss spectrum; to the presence of Raman gain in the Ti:Al<sub>2</sub>O<sub>3</sub> which produces an intensity dependent shift of the laser

frequency, that couples intensity and frequency fluctuations; and/or to incomplete inversion. Since the timing jitter due to quantum noise is not far from the measured value, large improvement of the timing jitter can be obtained only by means of active stabilization of the cavity [25] or by an active "retiming" of the pulse stream [22], [27].

## IX. CONCLUSION

We have presented an analysis of noise in passively mode-locked lasers with particular applicability to APM or KLM-locked lasers in which the Kerr nonlinearity would introduce an appreciable chirp unless compensated by negative GVD. In this case, the system obeys perturbed soliton equations. To the extent that (40) through (43) are dynamic equations of motion with a simple physical interpretation, our approach may predict noise behavior that is not strictly based on soliton perturbation theory. We consider both classical and quantum noise sources, generally  $\delta$  function correlated on the slow time scale of multiple resonator transit times. When the pulsewidth and timing fluctuations are uncorrelated, a set of symmetric spectral lines spaced by  $2\pi/T_R$  is predicted in the low frequency spectrum of the detector current. This is in general agreement with von der Linde's predictions. In the presence of correlation, these spectral lines may become asymmetric. The interference of timing jitter from the low frequency spectra, possible for actively mode locked systems, may lead to erroneous results when applied to the passively mode-locked case. The spectral lines of the photocurrent show a  $\Omega^{-4}$  dependence for  $\Omega > 1/\tau_p$ , where  $\tau_p$  is a critical time related to the gain bandwidth, (61). These predictions are, generally, in agreement with experimental observations.

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